Comments on the solution of the spherical Raman-Nath equation

1984 J. Phys. A: Math. Gen. 17 L395

(http://iopscience.iop.org/0305-4470/17/7/005)

J. Phys. A: Math. Gen. 17 (1984) L395–L398. Printed in Great Britain

LETTER TO THE EDITOR

Comments on the solution of the spherical Raman–Nath equation

P Bosco†, G Dattoli†‡ and M Richetta§||

† Physics Department and Quantum Institute, University of California at Santa Barbara, Santa Barbara, California 93106, USA
§ ENEA Dip TIB, Divisione Fisica Applicata, cp 65, 00044 Frascati, Rome, Italy

Received 31 January 1984

Abstract. We present a preliminary investigation of the spherical Raman–Nath equation, and discuss the connections between its solution and previously known cases.

As has already been pointed out in a number of previous papers (Dattoli and Renieri 1984, Bosco and Dattoli 1983, Ciocci *et al* 1984, Dattoli *et al* 1984), the Raman–Nath (RN) type of equations has become increasingly popular, because they are useful in describing a large number of physical phenomena. They also are topics from the purely mathematical point of view, because it has been shown (Dattoli *et al* 1984) how they may be useful for the solution of a class of difference equations. In this letter we would like to present a preliminary discussion of a new type of RN equations, which will be called, from now on, the spherical version (SRN), and are useful in analysing a few physical problems. Indeed they account for the time evolution of coherent Bloch states (Arecchi *et al* 1972) driven by external fields.

Before beginning the discussion of the problem, we recall that the Raman–Nath equations belong to a differential-difference class of equations. A very simplified version of the RN equations (Dattoli and Renieri 1984, Bosco and Dattoli 1983, Ciocci *et al* 1984, Dattoli *et al* 1984) can be given in the form

$$\mathrm{i}\, \mathrm{d}C_l/\mathrm{d}\tau = \Omega[C_{l+1} + C_{l-1}] \tag{1a}$$

$$\mathrm{i}\, \mathrm{d}C_l/\mathrm{d}\tau = \Omega[(l+1)^{1/2}C_{l+1} + l^{1/2}C_{l-1}] \tag{1b}$$

$$\mathrm{i}\, \mathrm{d}C_l/\mathrm{d}\tau = \Omega[(n+l+1)^{1/2}C_{l+1} + (n+l)^{1/2}C_{l-1}] \tag{1c}$$

with the initial conditions $C_l(0) = \delta_{l,0}$. Ω is a known constant and n is a positive integer. The solutions of equations $(1a)$–$(1c)$, found by means of the operatorial techniques developed in (Dattoli and Renieri 1984, Bosco and Dattoli 1983, Ciocci *et al* 1984 and Dattoli *et al* 1984), are respectively

$$C_l = (-\mathrm{i})^l J_l(2\Omega\tau) \tag{2a}$$

$$C_l = (-\mathrm{i})^l (l!)^{-1/2} \exp(-\tfrac{1}{2}\Omega^2\tau^2)(\Omega\tau)^l \tag{2b}$$

$$C_l = (-\mathrm{i})^l [n!/(n+l)!]^{1/2} (\Omega\tau)^l \exp(-\tfrac{1}{2}\Omega^2\tau^2) L_n^l[(\Omega\tau)^2] \tag{2c}$$

‡ Permanent address: ENEA, Dip TIB, Divisione Fisica Applicata, CRE Frascati, CP 65–00044 Frascati, Rome, Italy.
|| ENEA student.

where $J_l(x)$ is the lth Bessel function of the first kind, and $L_n^l(x)$ are the generalised Laguerre polynomials. Solution (2a) is a simple expression, in terms of Bessel functions. The second solution is written in terms of Charlier–Poisson polynomials (Szëgo 1959). The third solution is more complicated than the first two, which can, in fact, be considered special limiting cases of (2c) (Ciocci *et al* 1984, Dattoli *et al* 1984).

The equation we would like to discuss in this paper is in the form

$$\mathrm{i}\,\mathrm{d}C_l/\mathrm{d}\tau=\Omega\{[(l+1)(n_--l)]^{1/2}C_{l+1}+[l(n_--l+1)]^{1/2}C_{l-1}\} \tag{3}$$

with the initial conditions $C_l(0)=\delta_{l,0}$. The reason why we have named equation (3) an SRN equation will be made clear in the following.

In order to be consistent with the previous treatment, we can redefine the unknown functions $C_l(\tau)$ as follows

$$C_l(x)=(-\mathrm{i})^l M_l(x) \tag{4}$$

where $x=\Omega\tau$ and $M_l(0)=\mathrm{i}^l\delta_{l,0}$. Therefore, we have

$$\mathrm{d}M_l/\mathrm{d}x=\{-[(l+1)(n_--l)]^{1/2}M_{l+1}+[l(n_--l+1)]^{1/2}M_{l-1}\}. \tag{5}$$

The structure of this equation is complicated by products in the square roots, which are more involved than the products present in (2a)–(2c). One way of approaching (5) in a more useful fashion can be obtained by recalling the harmonic operators used in deriving (1b) and (1c), and by generalising that procedure. We can introduce coupled creation–annihilation operators (a_+^+, a_-^+) and (a_+, a_-), defined by their operation on the 'states' M_l:

$$a_+^+a_-M_l=[(l+1)(n_--l)]^{1/2}M_{l+1} \tag{6a}$$

$$a_-^+a_+M_l=[l(n_--l+1)]^{1/2}M_{l-1}. \tag{6b}$$

Equation (5) can now be expressed in the form

$$\mathrm{d}M_l/\mathrm{d}x=-[a_+^+a_--a_-^+a_+]M_l. \tag{7}$$

By introducing the previous set of coupled harmonic oscillators we have only apparently simplified the problem. We can, however, use Schwinger's technique (1965) to transform the coupled bosonic operators into angular momentum operators J_+, J_-, J_z, defined as

$$J_+=a_+^+a_- \qquad J_-=a_-^+a_+ \qquad J_z=\tfrac{1}{2}[a_+^+a_+-a_-^+a_-]. \tag{8}$$

Equation (7) can now be written in the form

$$\mathrm{d}M_l/\mathrm{d}x=-(J_+-J_-)M_l. \tag{9}$$

This procedure explains why equation (3) was named an SRN equation.

The formal solution to (8) is easily found to be

$$M_l(x)=\exp[-(J_+-J_-)x]M_l(0). \tag{10}$$

In order to present its explicit analytical solution, it is necessary to disentangle the exponentials by considering the ordering properties of the operators J_+, J_- and J_z. These operators are the generators of the 'simple split three-dimensional' Lie algebra, with commutators

$$[-J_-,J_+]=2J_z \qquad [-J_-,2J_z]=-2J_- \qquad [J_+,2J_z]=-2J_+. \tag{11}$$

Using Kirzhnits' procedure (8a)–(8b), we can disentangle the exponents in (10) and we obtain

$$\exp[-x(-J_-+J_+)]=\exp[g_1(x)J_+]\exp[2g_2(x)J_z]\exp[-g_3(x)J_-] \qquad (12)$$

where the three functions $g_i(x)$ are defined as

$$g_1(x)=g_3(x)=\tan x \qquad g_2(x)=-\ln\cos x. \qquad (13)$$

Expanding the exponentials, and using rules (6), we obtain

$$C_l(x)=(-\mathrm{i})^l\binom{n_-}{l}^{1/2}(\tan x)^l(\cos x)^{n_-} \qquad (14)$$

which is the binomial generalisation of the Poisson–Charlier polynomials.

In the very large n_- limit, the three-dimensional Lie algebra reduces to that of the creation–annihilation operators. It may be easily shown that in this limit

$$g_1(x)\equiv g_2(x)=-x \qquad g_2(x)=\tfrac{1}{2}x^2. \qquad (15)$$

Therefore, Kirzhnits' formula (8a) reduces to the well known Baker–Haussdorf formula, and, in the large n_- limit, we find

$$C_l(\tau)=(-\mathrm{i})^l(l!)^{-1/2}(\bar\Omega\tau)^l\exp[-\tfrac{1}{2}(\bar\Omega\tau)^2] \qquad (16)$$

where $\bar\Omega=\Omega(n_-)^{1/2}$. This is the same result as that shown in (1b). We have considered a special case, that may be interpreted as the interaction between a grounded oscillator and an excited oscillator. As a result of this interaction, the first oscillator is excited at the expense of the second one.

We will now examine the case in which both oscillators have already been excited. The corresponding differential equation is

$$\mathrm{d}C_l/\mathrm{d}\tau=\Omega\{[(n_++l+1)(n_--l)]^{1/2}C_{l+1}+[(n_++l)(n_--l+1)]^{1/2}C_{l-1}\}, \qquad (17)$$

can be obtained by using a direct generalisation of the procedures discussed earlier, and can be expressed in the form

$$C_l(x)=(-\mathrm{i})^l\left(\frac{(\tan x)^l}{(\cos x)^{n_+-n_-}}\right)\left(\frac{(n_++l)!\,n_+!}{(n_--l)!\,n_-!}\right)^{1/2}\sum_{s=0}^{\infty}\frac{(n_-+s)!}{(n_+-s)!}\frac{(-1)^s}{s!(l+s)!}(\sin x)^{2s}. \qquad (18)$$

It is straightforward to verify that, in the $n_+=0$ limit, (18) reduces to (14). On the other hand, in the very large n_- case, (14) can be expressed in the form

$$C_l\approx(-\mathrm{i})^l(\bar\Omega\tau)^l\exp[-\tfrac{1}{2}(\bar\Omega\tau)^2][n_+!/(n_++l)!]^{1/2}L^l_{n_+}[(\bar\Omega\tau)^2] \qquad (19)$$

i.e. solution (2c).

Taking the large n_+ limit in (19), and using Laguerre polynomials' asymptotic properties, we find

$$C_l\approx(-\mathrm{i})^lJ_l[2\bar{\bar\Omega}\tau] \qquad (20)$$

where $\bar{\bar\Omega}=(n_+n_-)^{1/2}\Omega$, which is precisely equation (2a). We have shown that (17) is the most general form of the equation considered in this letter. All the solutions discussed before are, in fact, a particular case of the solution of equation (17).

In a forthcoming paper we will discuss a more complicated form of the equation we have discussed, namely

$$\mathrm{i}\,\mathrm{d}C_l/\mathrm{d}\tau=(\alpha+\beta l)lC_l+\Omega\{[(l+1)(n_--l)]^{1/2}C_{l+1}+[l(n_--l+1)]^{1/2}C_{l-1}\} \qquad (21)$$

with the usual initial conditions $C_l(0) = \delta_{l,0}$. This equation is relevant to the analysis of the stimulated Compton scattering.

It is a pleasure to thank W B Colson, J Gallardo, A Renieri and C LoSurdo for the discussions we had with them on this problem. One of the authors (GD) would like to acknowledge the financial support from UCSB Quantum Institute, where this work was completed.

References

Arecchi F T, Courtens E, Gilmore G and Thomas H 1972 *Phys. Rev.* A **6** 2211
Bosco P and Dattoli G 1983 *J. Phys. A: Math. Gen.* **16** 4409
Ciocci F, Dattoli G and Richetta M 1984 *J. Phys. A: Math. Gen.* **17** 1333
Dattoli G and Renieri A 1984 *Theoretical and experimental aspects of the Free Electron Laser* in *Laser Handbook* 4th vol, ed M L Stitch and M S Bass (Amsterdam: North-Holland) to appear
Dattoli G, Richetta M and Pinto I 1984 *J. Phys. A: Math. Gen.* submitted
Kirzhnits D A 1967 *Field Theoretical Methods in Many-Body Systems* (Oxford: Pergamon)
Lebedev N N 1972 *Special Functions and their Applications* (New York: Dover)
Schwinger J 1965 *Quantum Theory of Angular Momentum* ed L C Biedenharn and H Van Dam (New York: Academic)
Szëgo G 1959 *Orthogonal Polynomials, Am. Math. Soc. Colloquium, Pub. 23* rev edn
Witschel M 1975 *J. Phys. A: Math. Gen.* **8** 143